e = 0.01, 0.010020000
θ_p = 60°, 60.118205°
ρ_1 = 779.80816, 779.80865
ρ_2 = 806.28246, 806.28279
ρ_3 = 833.64203, 833.64220
θ_1 = 100°, 99.999972°
θ_2 = 100.53°, 100.529970°
θ_3 = 101.07000°, 101.069968°

Case 3

i = 30°, 30.000009°
a = 16000, 16001.358
e = 0.30, 0.30005244
θ_p = −20°, −20.003833°
ρ_1 = 8113.3207, 8113.3793
ρ_2 = 8126.2608, 8126.3195
ρ_3 = 8139.9188, 8139.977
θ_1 = 10°, 9.99999°
θ_2 = 10.53°, 10.529991°
θ_3 = 11.07°, 11.069992°

Case 4

i = 30°, 29.999965°
a = 16000, 15998.423
e = 0.80, 0.79998041
θ_p = −20°, −20.00039°
ρ_1 = 41.443169, 41.442664
ρ_2 = 44.263139, 44.262486
ρ_3 = 65.061448, 65.060481
θ_1 = 10°, 10.000003°
θ_2 = 10.53°, 10.529995°
θ_3 = 11.07°, 11.069987°

Case 5

i = 30°, 29.999979°
a = −16000, −15998.314
e = 1.50, 1.5000575
ρ_1 = 12143.289, 12143.362
ρ_2 = 12169.763, 12169.836
ρ_3 = 12197.176, 12197.248
θ_1 = 10.0000°, 9.999996°
θ_2 = 10.5300°, 10.599995°
θ_3 = 11.0700°, 11.069995°

Case 6

i = 30°, 29.99990°
a = −16000, −15997.779 naut miles
e = 20.0000, 20.002653
ρ_1 = 351905.91, 351905.91 naut miles
ρ_2 = 353663.25, 353663.25 naut miles
ρ_3 = 355449.83, 355449.83 naut miles
θ_1 = 10.000°, 9.999992°
θ_2 = 10.5300°, 10.529993°
θ_3 = 11.0700°, 11.069992°

References

[1] Danby, J. M. A., *Fundamentals of Celestial Mechanics* (The Macmillan Co., New York, 1962); for the formulas of Gibbs, see p. 176, Eqs. (7.3.6) and (7.3.7).

[2] Herget, P., "The determination of orbits," *The Computation of Orbits* (University of Cincinnati, Cincinnati, Ohio, 1948).

Technical Comments

Heat Transfer at Zero Prandtl Number in Flows with Variable Thermal Properties

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Edwards and Tellep[1] have calculated the heat-transfer rate from a fluid with power-law thermal properties in the limit of zero Prandtl number. They solve only the energy equation, since they claim that the nonviscous momentum equation is satisfied by taking the component of velocity parallel to the wall (u) to be everywhere equal to its local external value $u_e(x)$. It is the purpose of this note to point out that this solution of the momentum equation is not correct unless u_e is a constant, so the heat-transfer rates calculated by Edwards and Tellep are good only for the "flat-plate" case. Furthermore, for this case, a simple approximate analytical expression can be derived for the heat-transfer rate, which can be fitted to the exact calculations (for a cold wall) within ±3%.

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In the von Mises variables x, ψ the nonviscous momentum equation is

$$u\,\frac{\partial u}{\partial x} - \frac{\rho_e}{\rho}\,u_e\,\frac{du_e}{dx} = 0 \tag{1}$$

It is clear that $u = u_e(x)$ is a solution only if $\rho = \rho_e(x)$ or if u_e is constant. The former case is the incompressible fluid previously treated by many authors, but does not correspond to the variable-property fluid treated by Edwards and Tellep. For variable properties, only u_e = const satisfies the momentum equation independent of the solution of the energy equation, and so it is only to this case that the work of Edwards and Tellep applies. For variable u_e and variable properties, the momentum and energy equations are coupled, and solution is made difficult, even for similarity cases, by a singularity at the wall. The physical reason for the coupling is easy to see. If the temperature varies in the boundary layer, so does the density, and that effects the momentum balance through the inertia terms, even if the viscous terms are neglected.

Edwards and Tellep's results are thus applicable to the flat-plate geometry in the zero Prandtl number limit. This is identical with the case recently studied by Jepson,[2] as well as the end wall case in a shock tube recently studied by Fay and Kemp.[3] Following an idea of Jepson,[2] the author has derived by analytical means an approximate expression for the heat-transfer rate to a cold wall which can be fitted very accurately to exact solutions.[4] This expression is, in the notation of Ref. 1,

$$q = [(u_e/2x)\rho_e k_e c_{pe}]^{1/2}\,(T_e - T_0)Q \tag{2}$$

$$Q \equiv \frac{1}{(2)^{1/2}}\,\frac{\rho_w k_w}{\rho_e k_e}\left(\frac{d}{d\eta}\,\frac{T - T_0}{T_e - T_0}\right)_w = \frac{\Omega'}{(2)^{1/2}\,(r + s + 1)} \tag{3}$$

where the general expression for Q is

$$Q = 1.13 \left[\int_{\theta_w}^{1} \frac{\rho k}{\rho_e k_e} d\theta \int^{1} \frac{c_p{}^*}{c_{pe}} d\theta^* \right]^{1/2} \qquad \theta \equiv \frac{T - T_0}{T_e - T_0} \tag{4}$$

For Edwards and Tellep's power-law properties

$$k/k_e = \theta^r \qquad \rho/\rho_e = \theta^s \qquad c_p/c_{pe} = \theta^t \tag{5}$$

the expression for Q becomes

$$Q = \frac{1.13}{(t+1)^{1/2}} \left[\frac{1 - \theta_w{}^{r+s+1}}{r+s+1} - \frac{1 - \theta_w{}^{r+s+t+2}}{r+s+t+2} \right]^{1/2} \tag{6}$$

Q is related to the Ω' of Ref. 1 by Eq. (3), and Ω is related to θ by

$$\Omega = \theta^{r+s+1} \tag{7}$$

Thus, for cold walls, Eq. (6) is a correlation formula for the curves of Fig. 1 of Ref. 1. The expression for Q has been found accurate to within 3% for $0 < \theta_w < 1$, $\frac{1}{2} < r < \frac{5}{2}$, $s = -1$, $t = 0$ and should be equally accurate for other values of s and t.

Reference 4 also contains expressions for upper and lower bounds on the exact value of Q, involving integrals similar to the ones given in Eq. (4), as well as an analysis for stagnation points at Prandtl numbers around unity.

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[2] Jepson, B-M., "Heat transfer in a completely ionized gas," Magnetogasdynamics Lab. Rept. 61-7, Dept. of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Mass. (July 1961).

[3] Fay, J. A. and Kemp, N. H., "Theory of end wall heat transfer in a monatomic gas, including ionization effects," Research Rept. 166, Avco-Everett Research Lab., Everett, Mass. (1963).

[4] Kemp, N. H., "Approximate analytical solution of similarity boundary layer equations with variable fluid properties," Publ. 64-6, Fluid Mechanics Lab., Dept. of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Mass. (1964).

Reply by Authors to N. H. Kemp

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In the preceding note Kemp makes the point that the solution to the momentum equation for inviscid flow is $u = u_e$ only when the fluid has constant density or when the flow is over a flat plate. The point is obvious and needs no laboring. Edwards and Tellep in their 1961 note[1] ruled out other cases with the statement "Only for the flat plate with $du_e/dx = 0$ and $dP/dx = 0$ will the analysis presented hold for fluids with pressure dependent properties." What was intended was to rule out the compressible fluid. The authors simply overlooked ruling out the hypothetical fluid which is incompressible but capable of significant thermal expansion or contraction. It is this hypothetical fluid that is the concern of Kemp, not to treat, but to eliminate from consideration.

An observation on the original note and the present one by Kemp is that the original exact results bear out the use of a reference temperature as suggested by Rubesin and Johnson[2] and Eckert.[3] Derivation of an expression for the reference temperature is facilitated by fitting the exact solution for $\Omega'(0)/(1 - \Omega_w)$ with $(2/\pi^{1/2})[(1 - am) + am\Omega_w]$, where a is adjusted for the best fit.[4] When T_w is not greatly different from T_e, a linearized result gives very nearly the result recommended by Eckert. The reference temperature concept gives results as useful as the approximate relation of Kemp or the previous exact results.

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[2] Rubesin, M. W. and Johnson, H. A., "A critical review of skin friction and heat transfer solutions of the laminar boundary layer on a flat plate," Trans. Am. Soc. Mech. Engrs. **71**, 383 (1949).

[3] Eckert, E. R. G., "Engineering relations for heat transfer and friction in high velocity laminar and turbulent boundary layer flow over a surface with constant pressure and temperature," Trans. Am. Soc. Mech. Engrs. **78**, 1273 (1956).

[4] Zwick, E., unpublished solution to homework problem (1962).

Comment on "Method for the Determination of Velocity Distribution in a Thin Liquid Film"

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In a recent note, Persson[1] described a method of measuring velocity profiles in thin films of liquid. This was done by photographing, with a finite exposure, small particles moving with the fluid. The velocities were obtained by measuring the length of the resultant images. The frequency with which various velocities were observed was determined, and hence, by assuming that the particles were uniformly distributed throughout the liquid, the velocity profile could be calculated.

Recent work by Jeffrey[2] has shown that in full tube Poiseuille flow there is a marked tendency for solid particles to migrate across the streamlines and take up a position at about one-third of the radius from the axis. Although this complete migration is admittedly slow, the region near the tube wall rapidly becomes deficient in particles.

There is no reason to assume that this effect does not occur in film flow. If it does, it will produce a particle-deficient layer near the wall, and so velocity profiles determined by Persson's method would have an error in the distance scale. This would be in such a direction as to give a positive velocity at the wall. Inspection of Persson's curves shows that this tendency does in fact exist, and that the unexpected kink in his velocity profiles arises because the profile has been forced to pass through $V = 0$ at $\delta/\delta_0 = 0$.

Some years ago the author[3–5] published an account of a method of measuring velocity profiles in thin films of liquid which avoids the difficulties inherent in Persson's method. Instead of photographing the particles with one camera only, two were used, inclined at an angle of about 40° to each other. The velocities were determined by the same method as Persson used, but the distances from the wall were found from the stereoscopic effect, i.e., the relative positions of the images on the two films. This technique is, in principle, the same as that used for the determination of altitude in aerial surveying. It was found possible by this technique to measure distances across the film to an accuracy of about ±0.001 cm,

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